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MIXED FINITE-DIFFERENCE SCHEME FOR FREE-VIBRATION ANALYSIS OF NONCIRCULAR CYLINDERS

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SUMMARY

A mixed finite-difference scheme is presented for the free-vibration analysis of simply supported closed noncircular cylindrical shells. The problem is formulated in terms of eight first-order differential equations in the circumferential coordinate which possess a symmetric coefficient matrix and are free of the derivatives of the elastic and geometric characteristics of the shell. In the finite-difference discretization, two interlacing grids are used for the different fundamental unknowns in such a way as to avoid averaging in the difference-quotient expressions used for the first derivative. The resulting finite-difference equations are symmetric. The inverse-power method is used for obtaining the eigenvalues and eigenvectors.

Numerical studies of the effects of reducing the local discretization error and of mesh refinement on the accuracy and convergence of the solutions obtained by the scheme developed, as well as by the conventional schemes, are discussed. Both oval and elliptic profiles with constant and variable thicknesses have been considered, and, in all cases, a monotonic convergence for the eigenvalues was obtained with the reduction of the local discretization error and/or the increase in the number of finite-difference intervals in the case of the modified scheme. Comparisons were also made between the results obtained from this study and results from some of the previous approximate analyses. It was found that the proposed scheme, in addition to a number of other advantages, leads to highly accurate results even when a small number of finite-difference intervals are used.

INTRODUCTION

Although considerable literature has been devoted to the free-vibration analysis of circular cylindrical shells, investigations of the vibrations of noncircular cylindrical shells are rather limited in extent. Cylindrical shells having a noncircular cross section

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are used in deep submersibles, aerospace vehicles, and other industrial applications, and, therefore, an understanding of their vibration characteristics is desirable. Because the radius of curvature of the noncircular cylinder varies with the circumferential coordinate, closed-form or analytic solutions cannot be obtained, in general, for this class of shells. Numerical or approximate techniques are necessary for their analysis.

Various approximate methods have been used for the free-vibration analysis of noncircular cylinders. Perturbation techniques were used by Klosner (refs. 1 and 2) for infinitely long cylinders and by Malkina (ref. 3) for shells with simply supported curved edges. Ivanyuta and Finkel'shteyn (ref. 4) applied the variational method of Oniashvili (ref. 5) in their study of free vibration of prestressed elliptic cylinders. Quite recently, Sewall et al. (refs. 6 and 7) applied the Rayleigh-Ritz technique to the vibration analysis of elliptic cylinders having simply supported, free, and clamped edges. Culberson and Boyd's (ref. 8) vibration analysis of simply supported oval cylinders involved a Fourier approach similar to that employed by Romano and Kempner (refs. 9 and 10) in the static stress analysis problem.

The present study was motivated by the approximate character of the solutions obtained, the lengthy algebraic manipulations involved, and convergence difficulties reported in the publications just cited. The purpose of this paper is to present a simple and accurate "mixed" finite-difference scheme for the determination of the natural frequencies and mode shapes of simply supported (longitudinal displacement is unrestrained), isotropic, closed arbitrary cylindrical shells with constant or variable wall thickness. The term "mixed" refers to the fact that both stress resultants and displacements are chosen as primary variables. The accuracy of the scheme is demonstrated by means of a number of numerical examples.

The analytical formulation is based on the linear Sanders-Budiansky first-approximation shell theory. (See ref. 11.) A Fourier approach is used to separate the variables, and the governing equations are reduced to eight first-order ordinary differential equations in the circumferential coordinate. The reduction to first-order differential equations is similar to that used by Goldberg (ref. 12), Kalnins (ref. 13), and others in conjunction with numerical integration techniques. In contrast to the derivations in references 12 and 13, however, the governing equations in the present study are arranged to yield a symmetric coefficient matrix for the differential equations. For cylindrical shells with variable geometric or elastic characteristics, the symmetry of the governing equations is a unique feature of this formulation and is particularly convenient when a finite-difference method is used for the solution, since the resulting finite-difference equations are also symmetric.

The finite-difference scheme presented herein is a modification of the method used in references 11 and 14 for the static analysis of the same class of shells. It is based on the use of two interlacing grids for the different fundamental unknowns in contrast to other methods, including references 12 and 13, in which all unknowns are specified at the same set of points. References 12 and 13 use numerical integration (and multisegment method). The first-order equation formulation simplifies the finite-difference discretization, and the use of interlacing grids results in reducing both the local discretization error and the bandwidth of the finite-difference field equations. In addition, the use of the interlacing grids results in a monotonic convergence for both the frequencies and the mode shapes with both reduction in local discretization error (through the use of higher order difference-quotient expressions) and mesh refinement. The cited advantages, coupled with the symmetric nature of the equations, lead to considerable improvement in the computational efficiency of the finite-difference scheme.

The formulation presented herein is limited to cases where the fundamental unknowns are separable with respect to the independent variables. This is the case for noncircular cylindrical shells having simply supported curved edges and mechanical characteristics independent of the longitudinal coordinate. The formulation can be applied to other boundary conditions through the use of Kantorovich's method in conjunction with the Hellinger-Reissner mixed variational principle (ref. 15). Although the presentation is focused on noncircular cylinders, the finite-difference scheme can be extended to other classes of structures.

SYMBOLS

a,b semimajor and semiminor axes of oval (or elliptic) profiles, respectively

C_o reference extensional rigidity of shell, chosen to be average extensional rigidity over length L_2, $\dfrac{Eh_o}{1 - \nu^2}$

c length ratio, L_2/L_1

$d \equiv \dfrac{d}{d\xi_2}$

E Young's modulus of elasticity

f circular frequency of vibration of shell, hertz

h local thickness of shell

h_o — reference thickness, chosen to be average thickness over length L_2

$i = \sqrt{-1}$

L_1, L_2 — axial and circumferential lengths of shell, respectively

l — finite-difference interval in ξ_2-direction

M_1, M_2, M_{12} — moment stress resultants

m — number of longitudinal half-waves

N — exponent of order of local discretization error

N_1, N_2, N_{12} — direct stress resultants

$\tilde{N}_{12}$ — modified (boundary) stress resultant

n — number of finite-difference intervals in ξ_2-direction in quarter of profile

$\bar{n}$ — number of circumferential waves

Q_1, Q_2 — transverse shear stress resultants

R — local radius of curvature of middle surface of shell

R_o — reference radius of curvature, chosen to be radius of a circle with same circumference as that of noncircular profile

t — time

u,v,w — displacement components of middle surface in coordinate directions

V_2 — effective (boundary) transverse shear stress resultant (see eq. (3b))

x_1, x_2, x_3 — curvilinear coordinates (see fig. 1)

$\bar{\alpha} = R_o/L_1$

$\beta = h_o/R_o$

δ amplitude of thickness variation, measured from average thickness over length L_2

γ measure of oval eccentricity

$\lambda = h/h_o$

ν Poisson's ratio

ω angular frequency, radians per second

ϕ_1, ϕ_2 rotation components of middle surface

ρ_s density of shell material

ρ curvature parameter, R_o/R

θ angle made by the normal at any point to the minor axis (see fig. 1)

ξ_α dimensionless coordinates on shell middle surface, $\xi_\alpha = \dfrac{x_\alpha}{L_\alpha}$ for $\alpha = 1,2$

$\lfloor\ \rceil$ row matrix

$\{\ \}$ column matrix

$[\ ]$ rectangular or square matrix

$\{\psi_m\}, \{H_m\}$ vectors of fundamental unknowns

Superscript T indicates transpose of a matrix.

MATHEMATICAL FORMULATION

Shell Geometry

The geometric characteristics of the shell are shown in figure 1 and are defined as follows: L_1 and L_2 are axial and circumferential lengths of the middle surface of the

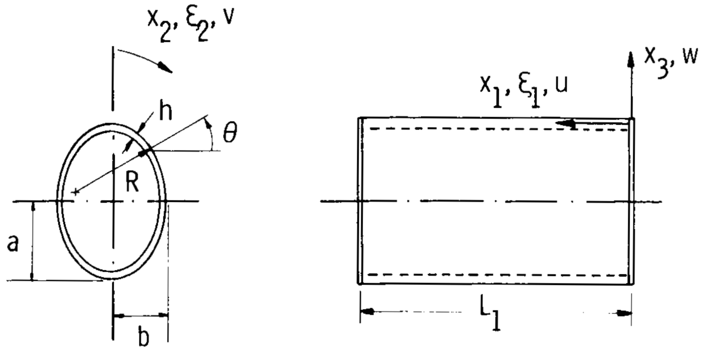


Figure 1.- Shell geometry.

shell; h and h_o are the local and reference shell thicknesses (h_o is chosen herein to be the average thickness of the shell over the length L_2); a and b are the semimajor and semiminor axes of the cross section.

In the present study, both oval and elliptic profiles are considered. The radius of curvature of the oval, in the form found in reference 9, is

$$R(x_2) = \frac{R_o}{1 + \gamma \cos 4\pi \dfrac{x_2}{L_2}} \tag{1}$$

where R_o is the radius of a circle having the same circumference as the oval and γ is an eccentricity parameter, which is a measure of the noncircularity of the cross section, $0 \leqq |\gamma| < 1$.

The radius of curvature of the elliptic profile is given by

$$R(x_2) = \frac{b^2}{a\left[1 - \left(1 - \dfrac{b^2}{a^2}\right)\cos^2\theta\right]^{3/2}} \tag{2}$$

where θ is the angle made by the normal at any point to the minor axis (fig. 1).

To cast the problem in nondimensional form, two dimensionless coordinates ξ_1 and ξ_2 are introduced, where

$$\xi_\alpha = \frac{x_\alpha}{L_\alpha} \qquad (\alpha = 1,2)$$

Also, the following dimensionless shell parameters are used:

$$\beta = \frac{h_o}{R_o}$$

$$\bar{\alpha} = \frac{R_o}{L_1}$$

$$c = \frac{L_2}{L_1} = 2\pi\bar{\alpha}$$

$$\rho(\xi_2) = \frac{R_o}{R}$$

$$\lambda(\xi_2) = \frac{h}{h_o}$$

Governing Differential Equations

The fundamental unknowns of the mixed formulation used herein are chosen to be the quantities that would appear in the statement of the boundary conditions along an edge parallel to the x_1-axis. For the Sanders-Budiansky first-approximation theory, these quantities are the four generalized displacements u, v, w, and ϕ_2 and the four stress resultants N_2, $\tilde{N}_{12}$, M_2, and V_2, where

$$\tilde{N}_{12} = N_{12} - \frac{M_{12}}{2R} \tag{3a}$$

$$V_2 = Q_2 + \frac{\partial M_{21}}{\partial x_1} \tag{3b}$$

Here N_{12} and M_{12} are the modified (symmetric) stress resultants used in the Sanders-Budiansky theory. (See ref. 16.)

If the different shell stress resultants and displacements are expanded in a Fourier series in the longitudinal coordinate so that the simple-support boundary conditions along the curved edges are satisfied, then

$$\lfloor v\ w\ \phi_2\ N_2\ M_2\ V_2 \rfloor = \sum_{m=1}^{\infty} \lfloor v_m\ w_m\ \phi_{2m}\ N_{2m}\ M_{2m}\ V_{2m} \rfloor \sin m\pi\xi_1\, e^{i\omega t} \tag{4}$$

and

$$\lfloor u\tilde{N}_{12}\ Q_1 \rfloor = \sum_{m=1}^{\infty} \lfloor u_m\ N_{12m}\ Q_{1m} \rfloor \cos m\pi\xi_1\, e^{i\omega t}$$

where ω is the angular frequency in radians/second.

The governing equations for arbitrary cylindrical shells then reduce to eight first-order ordinary differential equations for each Fourier harmonic. For the mth Fourier harmonic, these equations can be arranged to yield a symmetric coefficient matrix as follows (ref. 14):

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\left\{\begin{array}{c} H_m \\ \hline \psi_m \end{array}\right\} + \left[\begin{array}{c|c} 0 & I_1 \\ \hline I_1 & 0 \end{array}\right]\left\{\begin{array}{c} dH_m \\ \hline d\psi_m \end{array}\right\} + \omega^2\left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right]\left\{\begin{array}{c} H_m \\ \hline \psi_m \end{array}\right\} = 0 \tag{5}$$

where

$$d \equiv \frac{d}{d\xi_2}$$

$$\{H_m\}^T = \lfloor N_{2m}L_1/C_o \quad M_{2m}/C_o \quad u_m \quad w_m \rfloor$$

$$C_o = \frac{Eh_o}{1 - \nu^2}$$

$$\{\psi_m\}^T = \lfloor v_m \quad \phi_{2m}L_1 \quad N_{12m}L_1/C_o \quad V_{2m}L_1/C_o \rfloor$$

The $[A]$ and $[B]$ are 4 by 4 symmetric submatrices, $[I_1]$ is a diagonal submatrix, and $[m_1]$ and $[m_2]$ are diagonal mass matrices. The coefficients of these matrices are given in appendix A.

Equations (5) are free of the derivatives of elastic and geometric characteristics of the shell. Therefore, in cases where these quantities vary, no numerical approximation is needed, and, consequently, the overall accuracy of the solution should increase.

METHOD OF SOLUTION

Finite-Difference Discretization

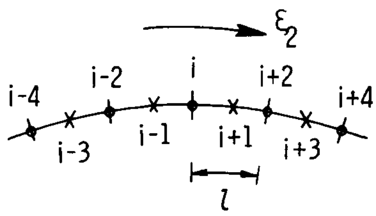
For the application of the finite-difference method, it is convenient to divide equations (5) into two groups and rewrite them in the following form:

$$[A]\left\{H_m\right\} + \left[I_1\right]\left\{d\psi_m\right\} = -\omega^2[m_1]\left\{H_m\right\} \tag{6a}$$

$$-[B]\left\{\psi_m\right\} - \left[I_1\right]\left\{dH_m\right\} = \omega^2[m_2]\left\{\psi_m\right\} \tag{6b}$$

The basic idea of the modified finite-difference scheme used in the present study is to define the first derivatives of each of the fundamental unknowns at points lying midway between the points of definition of the same unknowns. This can be accomplished by using two sets of interlacing grids for the two groups of fundamental unknowns $\left\{H_m\right\}$ and $\left\{\psi_m\right\}$.

The modified schemes to be used in conjunction with equations (6) are obtained by defining $\left\{\psi_m\right\}$ and $\left\{dH_m\right\}$ at points lying between the points of definition of $\left\{H_m\right\}$ and $\left\{d\psi_m\right\}$ as shown in figure 2. Consequently, the two groups of equations (eqs. (6a) and (6b)) are satisfied at different sets of points. The use of the above-mentioned interlacing grids makes it possible to avoid averaging in the difference-quotient expressions of the first derivative (ref. 17). Such averaging can lead to a significant reduction in accuracy as well as to nonmonotonic convergence. This will be discussed in the numerical examples.



∘ Control points for $\left\{H_m\right\}$ and $\left\{d\psi_m\right\}$

x Control points for $\left\{\psi_m\right\}$ and $\left\{dH_m\right\}$

Figure 2.- Control points for fundamental unknowns in modified scheme.

The finite-difference equations which simulate the governing equations (eqs. (6)) are obtained by replacing the first derivatives in these equations by their appropriate difference-quotient expressions. For evenly spaced intervals, the difference-quotient expressions with orders of local discretization error ranging between $O(l^2)$ and $O(l^{10})$, where l is the finite-difference interval, are given in appendix B for both the modified and the conventional schemes. Herein "by conventional schemes" means the schemes with all the fundamental unknowns and their derivatives defined at the same sets of points.

As an illustration, the l^4-modified finite-difference equations at generic interior points i and $i + 1$ are given by

$$[A]_i\left\{H_m\right\}_i + \frac{1}{l}\left[I_1\right]\left(\frac{1}{24}\left\{\psi_m\right\}_{i-3} - \frac{9}{8}\left\{\psi_m\right\}_{i-1} + \frac{9}{8}\left\{\psi_m\right\}_{i+1} - \frac{1}{24}\left\{\psi_m\right\}_{i+3}\right) = -\omega^2\left[m_1\right]_i\left\{H_m\right\}_i \tag{7a}$$

and

$$-[B]_{i+1}\left\{\psi_m\right\}_{i+1} - \frac{1}{l}\left[I_1\right]\left(\frac{1}{24}\left\{H_m\right\}_{i-2} - \frac{9}{8}\left\{H_m\right\}_i + \frac{9}{8}\left\{H_m\right\}_{i+2} - \frac{1}{24}\left\{H_m\right\}_{i+4}\right) = \omega^2\left[m_2\right]_{i+1}\left\{\psi_m\right\}_{i+1} \tag{7b}$$

The resulting finite-difference equations can be represented in the following compact form:

$$[K]\{Z\} = -\omega^2[\bar{m}]\{Z\} \tag{8}$$

where $[K]$ and $[\bar{m}]$ contain the "generalized" stiffness and mass distribution and $\{Z\}$ is the vector of unknowns composed of the subvectors $\left\{H_m\right\}_i$ and $\left\{\psi_m\right\}_j$ ($j = i$ and $i \pm 1$ in the conventional and the modified schemes, respectively) at the various finite-difference stations.

Equations (8) are banded and for this formulation will be symmetric in both the conventional and the modified schemes. The total bandwidth depends on the order of the approximation used and equals $8(N + 1) + 1$ and $8(N - 1) + 1$ in the conventional and the modified schemes (ref. 14), respectively, where N is the exponent of the local discretization error (appendix B).

Eigenvalue Extraction Technique

A variant of the inverse-power method with shifts similar to that presented in reference 18 has been used for the determination of the natural frequencies and mode shapes.

The method is described in detail in reference 18. In the present study, advantage was taken of the symmetric banded form of the matrix [K], and a direct Gaussian elimination procedure was used for each iteration to evaluate the new trial vector $\{Z\}$.

NUMERICAL STUDIES

Accuracy and Convergence Studies

Since the efficiency, among other factors, of the modified finite-difference scheme presented in this paper can best be assessed by its accuracy in the most unfavorable cases within the class of applicable problems and by the simplicity of its application, the scheme was applied to a large number of oval and elliptic cylindrical shells with increasing degree of noncircularity, having both uniform and variable thicknesses. The effects of reduction in local discretization error and of mesh refinement on the accuracy and rate of convergence of the natural frequencies and mode shapes from the modified scheme were studied and compared with those from the conventional schemes.

The class of shells considered in the present study included the simply supported elliptic shells analyzed by Sewall, Thompson, and Pusey (ref. 6) and the oval shells studied by Culberson and Boyd (ref. 8).

In addition, shells with variable thickness were studied, in which case the thickness variation was assumed to be of the form

$$h = h_o(1 + \delta \cos 4\pi \, \xi_2) \tag{9}$$

where δ represents the amplitude of thickness variation measured from the average thickness over the length L_2. All the shells analyzed had doubly symmetric profiles, and consequently, only one-quarter of the shell circumference was considered with the boundary conditions at the ends taken to be the symmetric or skew-symmetric conditions. The four different combinations of symmetry and skew symmetry of the mode shapes, along the axes of symmetry of the profile, were studied.

The local discretization error was reduced successively from $O(l^2)$ to $O(l^{10})$, and the number of finite-difference intervals in the shell quarter n was increased from 10 to 40. The reduction of local discretization error up to $O(l^{10})$ was feasible because of the extreme simplicity of the form of the governing differential equations (eqs. (6)). The numerical discretization was further simplified by the absence of boundary discretization error in the closed shells considered in the present study. The numerical results presented herein pertain to the minimum frequencies and some of the associated mode shapes.

Before conducting the convergence studies, the minimum natural frequencies for one and two longitudinal half-waves were obtained by the finite-difference method and compared with the analytical results from references 6 and 8 for 6061 aluminum cylinders having the following characteristics: $L_1 = 61.0$ cm (24 in.), $L_2 = 191.52$ cm (75.4 in.), $h = 0.0813$ cm (0.032 in.), $E = 68.95$ GN/m^2 (10^7 psi), $\nu = 0.3$, $\rho_s = 2768$ kg/m^3 (2.588×10^{-4} lb-sec^2/in^4). These characteristics were used throughout the numerical studies presented in this paper. Both the conventional and the modified solutions converged to the same frequency, and the converged results are summarized in table 1, where it is seen that the agreement between the finite-difference results and those of previous investigations is excellent. This agreement is particularly significant, since the finite-difference scheme presented herein is very simple to apply. This simplicity applies to the formulation of the problem, discretization, and computer implementation. Moreover, the difficulties with convergence as the degree of noncircularity increases (ref. 6, p. 14) or for higher Fourier harmonics (ref. 8, p. 1479) were, as is shown subsequently, totally nonexistent in the modified scheme presented herein.

To elaborate on this last point, the convergence studies of the minimum natural frequencies obtained for two shells with high degree of noncircularity are summarized in table 2. In addition, figures 3 to 5 provide some indication of the accuracy and rate of convergence of both the modified and the conventional schemes as applied to a representative problem of an oval shell with $\gamma = 0.70$ for three different modes. In figures 3 and 4, the rate of convergence with both reduction in local discretization error and mesh refinement is shown for the three modes. Figure 5 shows the mode shapes associated with the minimum frequency in the three cases.

The results of these numerical studies can be summarized as follows:

The natural frequencies and mode shapes obtained by the modified scheme were, in general, more accurate than those obtained by the corresponding conventional schemes having the same order of local discretization error and the same number of finite-difference intervals. In addition, the bandwidth of the resulting finite-difference equations is less in the modified scheme than in the corresponding conventional scheme. The increase in accuracy of the predictions of the modified scheme is mainly attributed to the use of "unaveraged" difference-quotient expressions. An indication of this accuracy is provided by the smaller error term in the first term of the remainder in the modified difference-quotient expressions (appendix B).

In general, the difference between the predictions of the modified and the conventional schemes decreases with the reduction of the local discretization error.

Although the convergence of the natural frequencies obtained by the modified scheme with both mesh refinement and reduction in local discretization error was monotonic, the convergence of the conventional schemes was, in some cases, nonmonotonic. (See figs. 3 and 4.)

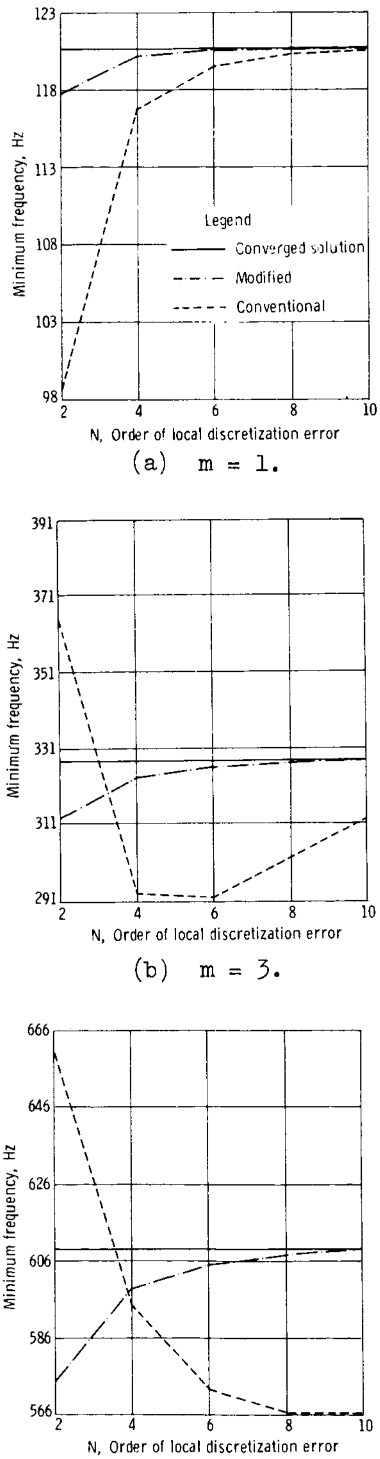


Figure 3.- Convergence of minimum frequency with reduction in local discretization error. Oval shell with γ = 0.70, n = 10.

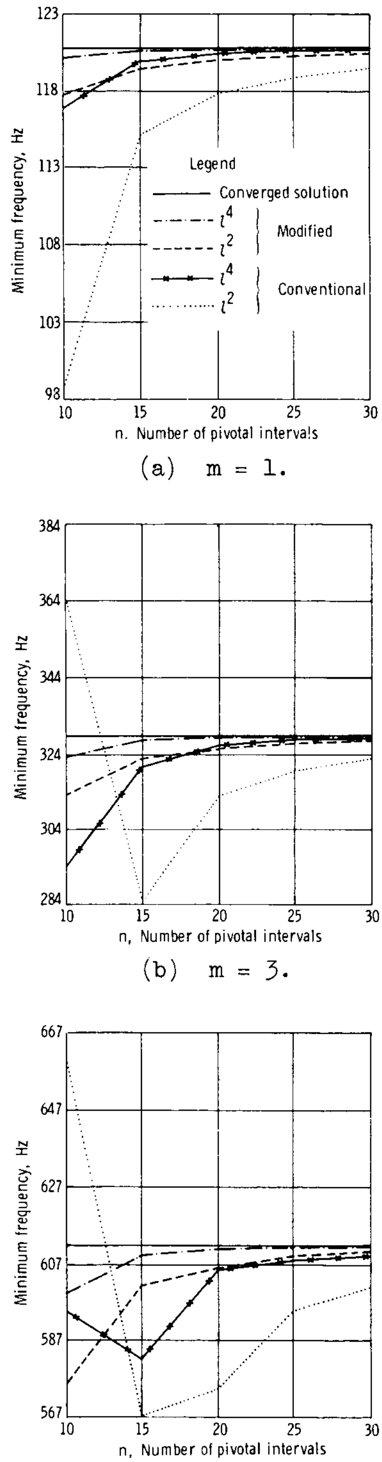


Figure 4.- Convergence of minimum frequency with mesh refinement. Oval shell with γ = 0.70.

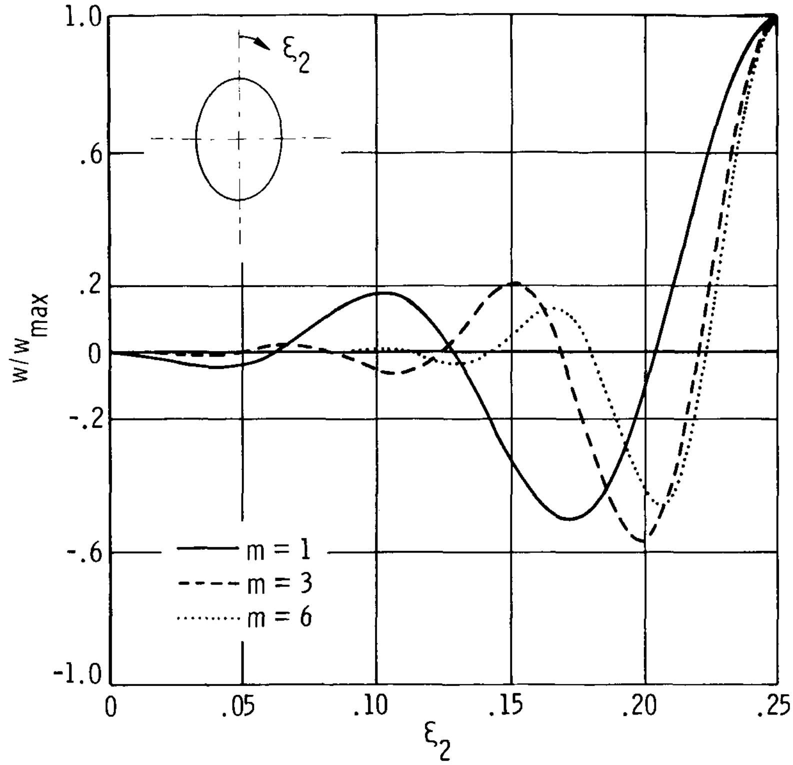


Figure 5.- Mode shapes associated with minimum frequency for oval shells with $\gamma = 0.70$.

For all the homogeneous shells analyzed, the frequencies obtained by use of the l^2-modified scheme were identical with those obtained by use of the l^2-conventional scheme with twice as many finite-difference intervals. This is not a coincidence, since the finite-difference equations for the l^2-modified scheme are identical with those of the corresponding conventional scheme with twice as many finite-difference intervals.

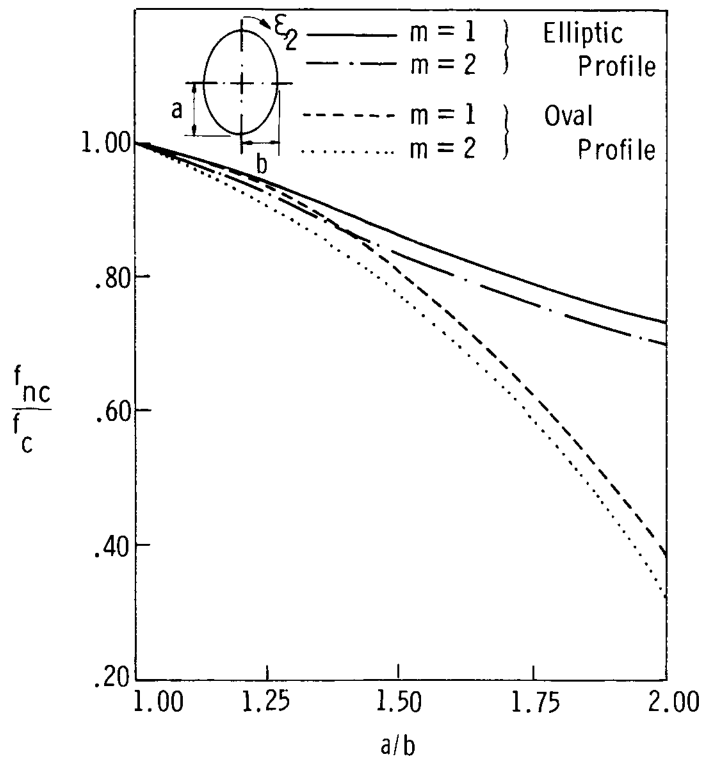
If the local discretization error is successively reduced from $O(l^2)$ to $O(l^{10})$, the gain in accuracy becomes less appreciable for each order of reduction, the more so the finer the mesh used. This is especially true for the modified schemes, where the gain in accuracy obtained by reducing the local discretization error beyond $O(l^6)$ is almost insignificant. Since the penalty of increasing the bandwidth for each order of reduction in the discretization error remains constant, the use of modified schemes with discretization error of order higher than l^6 is not recommended unless the algorithm used for handling the algebraic equations does not depend on the bandwidth.

Parametric Studies

After the reliability of the results obtained by the modified schemes had been established, a limited number of parametric studies were made to provide some insight into the effects of variations in the degree of noncircularity and local thickness on the minimum frequencies of the same group of oval and elliptic cylinders considered in the section entitled "Accuracy and Convergence Studies."

The ratios a/b (major to minor axis) and h_{max}/h_{min} (maximum to minimum shell thickness) were taken as the measures of noncircularity and thickness variation, respectively. The thickness variation was assumed to be given by equation (9). Such a thickness variation for a positive δ amounts to increasing the thickness in the shallow regions (ends of the minor axis) and decreasing it in the deep regions of the surface (ends of the major axis). The ratio h_{max}/h_{min} is given by $\frac{1 + \delta}{1 - \delta}$. The ratios a/b and h_{max}/h_{min} were varied between 1.0 and 2.0.

The effect of variation of a/b on the minimum frequencies of vibration is given in figure 6. It is shown that the minimum frequency of vibration decreases with the increase in a/b, that is, with the increase in the degree of noncircularity. This effect is more pronounced in oval shells than in elliptic shells having the same a/b ratio; this is especially true at higher values of a/b ($a/b \geqq 1.5$).



f_{nc} = Minimum frequency of the noncircular cylinder

f_c = Corresponding frequency of circular cylinder

Figure 6.- Effect of degree of noncircularity on minimum frequencies of simply supported cylinders for one and two longitudinal half-waves.

As an indication of the effect of variation in thickness on the minimum frequencies of vibration, table 3 gives the frequencies for both oval and elliptic cylinders having $a/b = 2$ for different values of h_{max}/h_{min}. The results presented in this table show that the chosen thickness variation resulted in an increase in the minimum frequency with increase of the ratio of h_{max}/h_{min}. The increase in frequency was somewhat larger for oval shells than for elliptical shells. Neither, however, changes the minimum frequency by more than about 12 percent.

CONCLUDING REMARKS

In this paper a modified mixed finite-difference scheme has been presented for the free-vibration analysis of simply supported closed arbitrary cylindrical shells with uniform and variable wall thickness. The scheme is based on the linear Sanders-Budiansky theory, and the governing equations consist of eight first-order ordinary differential equations in the circumferential coordinate. In the finite-difference discretization, two interlacing grids are used for the different fundamental unknowns in such a way as to reduce both the local discretization error and the bandwidth of the resulting finite-difference field equations. The inverse-power method is used to determine the eigenvalues and eigenvectors.

Numerical studies were made of the effects of reducing the local discretization error (from order $O(l^2)$ to order $O(l^{10})$ where l is the finite-difference interval) and of mesh refinement on the accuracy and convergence of solutions obtained by the modified scheme presented herein, as well as by the conventional scheme (with all the fundamental unknowns defined at the same set of points). In addition, parametric studies were made of the effects of variation of the degree of noncircularity and local thickness on the minimum frequencies of oval and elliptic cylinders. These parametric studies show that the reduction in the minimum frequency of vibration with the increase in the degree of noncircularity is more pronounced in oval shells than in elliptic shells having the same major to minor axis ratio. Also, a slight increase in the minimum frequency can be obtained by means of a redistribution of the shell thickness.

The finite-difference scheme presented herein is shown to combine a number of advantages over other finite-difference schemes previously reported in the literature. These advantages include the simplicity of the form of the governing differential equations, the absence of the derivatives of elastic and geometric characteristics in these equations, and the symmetry of their coefficient matrix. Although the first two advantages result from the use of first-order equation formulation and are independent of the method of discretization, the symmetry of the coefficient matrix can only be efficiently utilized in a finite-difference discretization. The cited advantages result in simplifications of the finite-difference discretization and provide flexibility for improving the accuracy of the scheme without undue complications. Moreover, the effort required in the computer implementation of the scheme (coding, debugging, and verification) can be expected to be significantly less than that required for other finite-difference schemes.

The numerical studies demonstrate the high accuracy of the predictions of the modified scheme, even when a small number of finite-difference intervals are used, and the monotonic character of the convergence of the natural frequencies with both reduction in local discretization error and mesh refinement.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., December 14, 1972.

APPENDIX A

FORMULAS FOR COEFFICIENTS IN THE GOVERNING EQUATIONS

The independent nonzero terms of the submatrices $[A]$ and $[B]$ in equations (5) are given as

$$A_{11} = -\frac{c}{\lambda}$$

$$A_{13} = -\nu m\pi c$$

$$A_{14} = \frac{c\rho}{\bar{\alpha}}$$

$$A_{22} = \frac{-12c}{\lambda^3\bar{\alpha}^2\beta^2}$$

$$A_{24} = \nu m^2\pi^2 c$$

$$A_{33} = m^2\pi^2 c\lambda(1 - \nu^2)$$

$$A_{44} = \frac{m^4\pi^4 c\lambda^3\bar{\alpha}^2\beta^2(1 - \nu^2)}{12}$$

and

$$B_{13} = -m\pi c$$

$$B_{14} = \frac{c\rho}{\bar{\alpha}}$$

$$B_{22} = -\frac{m^2\pi^2\bar{\alpha}^2\beta^2\lambda^3 c(1 - \nu)}{6\left(1 + \frac{\beta^2\lambda^2\rho^2}{48}\right)}$$

$$B_{23} = \frac{m\pi\bar{\alpha}\rho\lambda^2\beta^2 c}{12\left(1 + \frac{\beta^2\lambda^2\rho^2}{48}\right)}$$

$$B_{24} = -c$$

$$B_{33} = \frac{2c}{\lambda(1 - \nu)\left(1 + \frac{\beta^2\lambda^2\rho^2}{48}\right)}$$

$\left[I_1\right]$ is a diagonal submatrix given by

$$\left[I_1\right] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$[0]$ is a 4 by 4 null submatrix, and $\left[m_1\right]$ and $\left[m_2\right]$ are 4 by 4 diagonal mass submatrices given by

$$\left[m_1\right] = \kappa\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \left[m_2\right] = \kappa\begin{bmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

where $\kappa = -\frac{\rho_s \lambda h_o c L_1^2}{C_o}$ and ρ_s is the mass density.

DIFFERENCE-QUOTIENT EXPRESSIONS USED IN PRESENT STUDY

For evenly spaced intervals, the difference-quotient expressions at interior points for the conventional and the modified schemes can be written in the following compact form (fig. 7):

$$dF\Big|_i \approx \frac{1}{l} \sum_q a_{\pm q} F_{i \pm q} + E_R \tag{B1}$$

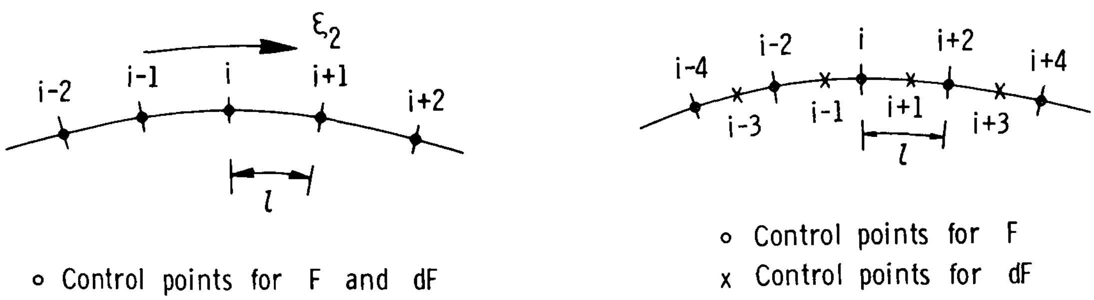
where $q = 0,1,2,3,\ldots$ for the conventional scheme, and

$$dF\Big|_i \approx \frac{1}{l} \sum_q a_{\pm(1+q)/2}\, F_{i \pm q} + E_R \tag{B2}$$

where $q = 1,3,5,7,\ldots$ for the modified scheme. In equations (B1) and (B2), F denotes any of the fundamental unknowns $\{H_m\}$ or $\{\psi_m\}$; a_r denotes weighting factors with $a_{-r} = -a_r$ and $a_0 = 0$; l is the finite-difference interval; E_R is the first term in the error series and is given by (ref. 19)

$$E_R = K_R\, l^N \left.\frac{d^{N+1}F}{dx^{N+1}}\right|_i$$

where l^N is the order of approximation and K_R is a multiplier. The $\pm$ sign before



(a) Conventional schemes.

(b) Modified schemes.

Figure 7.- Control points for F and dF in conventional and modified schemes.

a term denotes that the summation extends over the positive as well as the negative values of the index. In each term the sign of the two indices of a and F should be identical.

The different values of a_r and K_R for both the conventional and the modified schemes with local discretization errors up to $O(l^{10})$ are given in table 4.

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TABLE 1.- COMPARISON OF MINIMUM FREQUENCIES OBTAINED BY FINITE-DIFFERENCE METHOD WITH THOSE OBTAINED BY PREVIOUS INVESTIGATORS

m	$\bar{n}$	a/b	f, Hz, for elliptic profile		f, Hz, for oval profile	
			Reference 6	Finite differences	Reference 8	Finite differences
1	7	1.0	162.2	162.2	163.5	162.2
	7	1.176	[a]157.0	157.0	157.8	156.7
	7	1.538	138.5	138.6	129.6	129.9
	6	2.5	106.8	106.8	----	----
2	10	1.0	325.1	325.7	327.1	325.7
	10	1.176	310.6	310.7	310.4	309.3
	10	1.538	268.1	268.2	245.7	245.4
	9	2.5	202.4	202.3	----	----

[a]Skew-symmetric mode, all others are doubly symmetric mode.

TABLE 2.- CONVERGENCE OF MINIMUM FREQUENCIES OBTAINED BY DIFFERENT FINITE-DIFFERENCE SCHEMES

(a) Oval shell with $\frac{a}{b} = 1.538$ ($\gamma = -0.62706$), m = 2

n	Frequency, Hz, for –				
	l^2	l^4	l^6	l^8	l^{10}
Modified schemes					
10	236.1	242.8	244.5	245.0	245.2
15	241.7	244.9	245.3	[a]245.4	[a]245.4
20	243.4	245.2	[a]245.4		
25	244.1	245.3			
30	244.5	[a]245.4			
Conventional schemes					
10	256.5	214.8	229.6	240.1	243.1
15	220.7	241.5	244.5	245.2	245.3
20	236.1	256.1	245.2	[a]245.4	[a]245.4
25	239.9	244.9	[a]245.4		
30	241.7	245.2			

[a]Converged solution.

TABLE 2.- CONVERGENCE OF MINIMUM FREQUENCIES OBTAINED BY DIFFERENT FINITE-DIFFERENCE SCHEMES – Concluded

(b) Elliptic cylinder with $\frac{a}{b} = 1.538$, $m = 3$

n	Frequency, Hz, for –				
	l^2	l^4	l^6	l^8	l^{10}
Modified schemes					
10	380.9	389.5	392.2	393.4	393.9
15	389.4	393.5	394.3	394.5	394.5
20	391.8	394.2	394.5	[a]394.6	[a]394.6
25	392.8	394.4	394.5		
30	393.3	394.5	394.5		
Conventional schemes					
10	563.8	406.5	375.9	374.5	380.6
15	374.9	386.6	392.1	393.7	394.3
20	380.9	392.2	394.1	394.5	394.5
25	386.8	393.6	394.4	394.5	[a]394.6
30	389.4	394.1	394.5	394.5	

[a]Converged solution.

TABLE 3.- EFFECT OF THICKNESS VARIATION ON MINIMUM FREQUENCIES IN OVAL AND ELLIPTIC CYLINDERS WITH $\frac{a}{b} = 2$, $m = 2$

h_{max}/h_{min}	Minimum frequency, Hz	
	Elliptic profile	Oval profile
1.0	230.1	107.3
1.25	237.6	112.0
1.50	242.9	115.5
1.75	246.9	118.4
2.0	250.0	120.7

TABLE 4.- COEFFICIENTS IN THE FINITE-DIFFERENCE-QUOTIENT EXPRESSIONS

Coefficient	Order of approximation				
	l^2	l^4	l^6	l^8	l^{10}
Modified schemes					
a_1	1	9/8	75/64	1225/1024	19 845/16 384
a_2		-1/24	-25/384	-245/3072	-735/8192
a_3			3/640	49/5120	567/40 960
a_4				-5/7168	-1215/688 128
a_5					35/294 912
K_R	-1/24 (-4.167×10^{-2})	3/640 (4.687×10^{-3})	-5/7168 (-6.975×10^{-4})	35/294 912 (1.187×10^{-4})	-63/2 883 584 (-2.185×10^{-5})
Conventional schemes					
a_1	1/2	2/3	3/4	4/5	5/6
a_2		-1/12	-3/20	-1/5	-5/21
a_3			1/60	4/105	5/84
a_4				-1/280	-5/504
a_5					1/1260
K_R	-1/6 (-1.667×10^{-1})	1/30 (3.333×10^{-2})	-1/140 (-7.143×10^{-3})	1/630 (1.587×10^{-3})	-1/2772 (-3.608×10^{-4})

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16. Abstract

A mixed finite-difference scheme is presented for the free-vibration analysis of simply supported closed noncircular cylindrical shells. The problem is formulated in terms of eight first-order differential equations in the circumferential coordinate which possess a symmetric coefficient matrix and are free of the derivatives of the elastic and geometric characteristics of the shell. In the finite-difference discretization, two interlacing grids are used for the different fundamental unknowns in such a way as to avoid averaging in the difference-quotient expressions used for the first derivative. The resulting finite-difference equations are symmetric. The inverse-power method is used for obtaining the eigenvalues and eigenvectors.

Numerical studies of the effects of reducing the local discretization error and of mesh refinement on the accuracy and convergence of the solutions obtained by the scheme developed, as well as by the conventional schemes, are discussed. Both oval and elliptic profiles with constant and variable thicknesses have been considered, and, in all cases, a monotonic convergence for the eigenvalues was obtained with the reduction of the local discretization error and/or the increase in the number of finite-difference intervals in the case of the modified scheme. Comparisons were also made between the results obtained from this study and results from some of the previous approximate analyses. It was found that the proposed scheme, in addition to a number of other advantages, leads to highly accurate results even when a small number of finite-difference intervals are used.

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